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EXISTENCE OF A LOCAL SOLUTION OF A PARABOLIC – HYPERBOLIC FREE BOUNDARY PROBLEM

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ABSTRACT A parabolic-hyperbolic free boundary problem has been studied. After the study, we transform the problem for moving domain into an equivalent one which defined on a fixed domain. The existence and uniqueness of a local solution of the transformed problem by applying Banach fixed point theorem is derived.

KEYWORDS : Free boundary problem, local solution, moving domain, fixed domain

1.INTRODUCTION

In this article, we study a parabolic-hyperbolic free boundary problem[3] is studied. For this, Fick's law is assumed, i.e., (k_1K_P(C)P + (k_2K_Q(C)Q)C. Hence, C satisfies the following equation:

$$\frac{\partial c}{\partial t} = D_1\Delta C - \left(k_1K_P(C)\right)P + k_2K_Q(C)Q)C \text{ in } \Omega(t) \qquad (1.1)$$

$$C(x,t) = \bar{C} \text{ on } \partial\Omega(t),\ C(x,0) = C_0(x) \text{ in } \Omega(0), \qquad (1.2)$$

where (t) represents the domain at time t,
D[1] is positive constant. k_1 and k_2 are two positive constants, C is a positive constant.

Fick's law is also assumed $\left(\mu_1G_1(W)P + \mu_2G_2(W)Q\right)W$ is the drug consumption rate μ_1,μ_2 are two positive constants.

Hence, W satisfies $\frac{\partial W}{\partial t} = D_2\Delta W - \left(\mu_1G_1(W)P + \mu_2G_2(W)Q\right)W$ in Ω(t), in (t), (1.3)

$$W(x,t) = \bar{W} \text{ on } \partial\,\Omega(t), W(x,0) = W_0(x) \text{ in } \Omega(0), \qquad (1.4)$$

Where D_2 is to be a positive constant, (W) ιs a positive constant.

we denote v is the velocity fields v. assume by Darcy's law, we have

$$\underline{v} = -\nabla\sigma \text{ in } \Omega(t),\ t > 0, \qquad (1.5)$$

where σ is the pressure
P+Q+D=N in Ω(t),t>0, (1.6)

where N is a total number of cells per unit volume.
The mass conservation law for P,Q,D in

$$\frac{\partial P}{\partial t} + div(P\bar{v}) = [K_B(C) - K_Q(C) - K_A(C)]P + QK_P(C) - t_1G_1(W)P \quad \text{in } \Omega(t), t > 0 \qquad (1.7)$$

$$\frac{\partial Q}{\partial t} + div(Q\bar{v}) = K_Q(C)P - [K_P(C) + K_D(C)]Q - t_2G_2(W)Q \quad \text{in } \Omega(t), t > 0 \qquad (1.8)$$

$$\frac{\partial D}{\partial t} + div(D\bar{v}) = K_A(C)P + K_D(C)Q - K_RD + t_1G_1(W)P + t_2G_2(W)Q$$

$$\Omega(t), t > 0 \qquad (1.9)$$

where t_1 and t_2 are the positive constants.
We take the boundary conditions for σ to be

$$\sigma = \theta k \text{ on } \partial\Omega(t), t>0 \qquad (1.10)$$

$$\frac{\partial\sigma}{\partial n} = -V_n \text{ on } \partial\Omega(t), t > 0 \qquad (1.11)$$

and the initial data

$$P(x,0) = P_0(x), Q(x,0) = Q_0(x), D(x,0) = D_0(x)$$

for $x \in \Omega(0)$

where Ω(0) is given, θ is the surface tension, k is the mean curvature of the tumor surface $\frac{\partial}{\partial n}$ is the derivatives in the direction n of the outward normal, and Vn is the velocity of the free boundary ∂Ω(t) in the direction n.

Equation (1.10) is based on the assumption that the pressure σ on the surface of the tumor is proportional to the surface tension and (1.11) is a standard kinetic condition.

In this article, we consider spherically symmetric solution for the system[2] (1.1) – (1.12).

It is clear that, under the condition of spherical symmetry, for given $\bar{v}$ and R(t), σ we easily solved from (1.5) and (1.10).

It is obvious that from (1.7)-(1.9), we get the following equation for $\bar{v}$ By applying the L_p theory of parabolic equations, the characteristic theory of hyperbolic equations and the Banach fixed point theorem, we prove that there exists a unique local solution of (1.1) – (1.12). If we make an addition to (1.7) – (1.9), then we get the following equation for $\bar{V}$.

$$\frac{\partial P}{\partial t} + div(P\bar{v}) + \frac{\partial Q}{\partial t} + div(Q\bar{v}) + \frac{\partial D}{\partial t} + div(D\bar{v}) =$$

$$PK_B(C) - K_A(C)P + K_P(C)Q - t_1G_1(W)P + K_Q(C)P - K_P(C)Q - K_D(C)Q -$$

$$t_2G_2(W)Q + K_A(C) + K_D(C)Q - K_R(D) + t_1G_1(W)P + t_2G_2(W)Q$$

$$\frac{\partial}{\partial t}(P + Q + D) + div(P + Q + D)\bar{v} = PK_B(C) - K_R(D)$$

$$\frac{\partial}{\partial t}(N) + \nabla\bar{v}\,(N) = PK_B(C) - K_R(D)$$

$$\left(0 + div(\bar{v})\right)N = PK_B(C) - K_R(D)$$

$$N\left(div(\bar{v})\right) = PK_B(C) - K_R(D)$$

$$\left(div(\bar{v})\right) = \frac{1}{N}(PK_B(C) - K_R(D)) \qquad (1.13)$$

for $x \in \Omega(t)$, t>0 .

Conversely, from (1.13) and (1.7)-(1.9) we have

$$\frac{\partial}{\partial t}(P + Q + D) + div(P + Q + D)\bar{v} = \frac{1}{N}\left(PK_B(C) - K_R(D)\right) \times \left(N - (P + Q + D)\right) \quad \text{for } x \in \Omega(t), t>0 . \qquad (1.14)$$

By uniqueness, we deduce that (1.6) is equivalent to (1.13) and we use (1.13) instead of (1.6).

In this article the model[1] (1.1) – (1.12) is a three-dimensional model. Consider the well-posedness of this problem[6] under the case where the initial data and the solution are spherically symmetric. Hence, C, W, P, Q and D are spherically symmetric in the space variable, let r = |x|, we denote

$$C = C(r,t), W = W(r,t), P = P(r,t), Q = Q(r,t), D = D(r,t)$$

$$\text{for } 0 \le r \le R(t), t \ge 0, \text{ and}$$

$$C = C_0(r), W_0 = W_0(r), P_0 = P_0(r), Q_0 = Q_0(r), D_0 = D_0(r) \ \text{ for } 0 \le r \le R_0 = R(0)$$

We also assume that there is a scalar function[10] $V = V(r,t)$ such that $\overline{V} = (r,t)\frac{x}{R}$,

since σ is spherically symmetric in the space variable, as mentioned before, we eliminate the pressure and derive the model (1.1) – (1.12) as:

$$\frac{\partial C}{\partial t} = D_1 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial C}{\partial r}\right) - F(C,P,Q)C \ \text{ for } 0 < r < R(t), t > 0,$$

.15)

$$\frac{\partial C}{\partial r}(r,t) = 0 \text{ at } r = 0, C(r,t) = \bar{C} \text{ at } r = R(t) \text{ for } t > 0,$$

$$C(r,0) = C_0(r) \text{ for } 0 \le r \le R_0 \qquad .16)$$

$$\frac{\partial W}{\partial t} = D_1 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial W}{\partial r}\right) - G(W,P,Q)W \text{ for } 0 < r < R(t), t > 0, \qquad .17)$$

$$\frac{\partial W}{\partial r}(r,t) = 0 \text{ at } r = 0, W(r,t) = \overline{W} \text{at } r = R(t) \text{ for } t > 0, \qquad 1.18$$

(1.19)

$$W(r,0) = W_0(r) \text{ for } 0 \le r \le R_0 \qquad (1.20)$$

$$\frac{\partial P}{\partial t} + V\frac{\partial P}{\partial r} = g_{11}(C,W,P,Q,D)P + g_{12}(C,W,P,Q,D)Q + g_{13}(C,W,P,Q,D)D$$

$$\text{for} 0 \le r \le R(t), t > 0 \qquad (1.21)$$

$$\frac{\partial Q}{\partial t} + V\frac{\partial Q}{\partial r} = g_{21}(C,W,P,Q,D)P + g_{22}(C,W,P,Q,D)Q + g_{23}(C,W,P,Q,D)D$$

$$\text{for} 0 \le r \le R(t), t > 0 \qquad (1.22)$$

$$\frac{\partial D}{\partial t} + V\frac{\partial D}{\partial r} = g_{31}(C,W,P,Q,D)P + g_{32}(C,W,P,Q,D)Q + g_{33}(C,W,P,Q,D)D$$

$$\text{for } 0 \le r \le R(t), t > 0 \qquad (1.23)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V) = h(C,W,P,Q,D) \text{ for } 0 < r \le R(t), t > 0, \qquad (1.24)$$

$$V(0,t) = 0 \ \text{ for } t > 0. \qquad (1.25)$$

$$\frac{dR(t)}{dt} = V(R(t),t) \qquad \text{for} > 0, \qquad (1.26)$$

$$P(r,0) = P_0(r), Q(r,0) = Q_0(r), D(r,0) = D_0(r) \text{ for } 0 \le r \le R_0, \qquad (1.27)$$

$R(0) = R_0$ is prescribed,

where

$$F(C,P,Q) = k_1 K_P(C)P + k_2 K_Q(C)Q,$$

$$G(W,P,Q) = \mu_1 G_1(W)P + \mu_2 G_2(W)Q.$$

$$g_{11}(C,W,P,Q,D) = [K_B(C) - K_Q(C) - K_A(C) - t_1 G_1(W)] - \frac{1}{N}[K_B(C)P - K_R D],$$

$$g_{12}(C,W,P,Q,D) = K_P(C),$$

$$g_{13}(C,W,P,Q,D) = 0,$$

$$g_{21}(C,W,P,Q,D) = K_Q(C),$$

$$g_{22}(C,W,P,Q,D) = -[K_P(C) + K_D(C) + t_2 G_2(W)] - \frac{1}{N}[K_B(C)P - K_R D],$$

$$g_{23}(C,W,P,Q,D) = 0$$

$$g_{31}(C,W,P,Q,D) = K_A(C) + t_1 G_1(W),$$

$$g_{32}(C,W,P,Q,D) = K_D(C) + t_2 G_2(W),$$

$$g_{33}(C,W,P,Q,D) = -K_R - \frac{1}{N}[K_B(C)P - K_R D],$$

$$h(C,W,P,Q,D) = \frac{1}{N}[K_B(C)P - K_R D] \qquad (2.1)$$

SECTION-2 REFORMULATION OF THE PROBLEM

To transform the varying domain $\{(x,t): |x| - r < R(t), t > 0\}$ into a fixed domain, assume (R,C,W,P,Q,D) is a solution of (1.15)-(1.27) and R(t)>0 (t≥0), and make the change of variables,

$$\rho = \frac{r}{R(t)}, \tau = \int_0^t \frac{ds}{R^2(t)}, \eta(\tau) = R(t), \ c(\rho,\tau) = C(r,t), w(\rho,\tau) = W(r,t),$$

$$p(\rho,\tau) = P(r,t), q(\rho,\tau) = Q(r,t), d(\rho,\tau) = D(r,t),$$

$$u(\rho,\tau) = R(t)v(r,t),$$

then the free boundary problem (1.15) (1.27) is transformed into the initial-boundary value problem[2] on the fixed domain {(τ,ρ):0≤ρ≤1,τ≥0}

$$\frac{\partial c}{\partial \tau} = D_1 \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial c}{\partial \rho}\right) + u(1,\tau)\rho\frac{\partial c}{\partial \rho} - \eta^2 f(c,p,q)c \quad \text{for } 0 < \rho < 1, \ \tau > 0, \qquad (2.2)$$

$$\frac{\partial c}{\partial \rho}(0,\tau) = 0, c(1,\tau) = \bar{c} \qquad \text{for} > 0, \qquad (2.3)$$

$$c(\rho,0) = c_0(\rho) \qquad \text{for } 0 \le \rho \le 1, \qquad (2.4)$$

$$\frac{\partial w}{\partial \tau} = D_2 \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial w}{\partial \rho}\right) + u(1,\tau)\rho\frac{\partial w}{\partial \rho} - \eta^2 g(w,p,q)w \quad \text{for } 0 < \rho < 1, \ \tau > 0,$$

)

$$\frac{\partial w}{\partial \rho}(0,\tau) = 0, w(1,\tau) = \overline{w} \qquad \text{for } \tau > 0,$$

)

$$w(\rho,0) = w_0(\rho) \quad \text{for} 0 \le \rho \le 1,$$

)

$$\frac{\partial p}{\partial \tau} + v\frac{\partial p}{\partial \rho} = \eta^2[g_{11}(c,w,p,q,d)p + g_{12}(c,w,p,q,d)q + g_{13}(c,w,p,q,d)d] \qquad (2.8)$$

$$\text{for } \ 0 \le \rho \le 1, \tau > 0,$$

$$\frac{\partial q}{\partial \tau} + v\frac{\partial q}{\partial \rho} = \eta^2[g_{21}(c,w,p,q,d)p + g_{22}(c,w,p,q,d)q + g_{23}(c,w,p,q,d)d] \qquad (2.9)$$

$$\text{for } \ 0 \le \rho \le 1, \tau > 0,$$

(2.10)

$$\frac{\partial d}{\partial \tau} + v\frac{\partial d}{\partial \rho} = \eta^2[g_{31}(c,w,p,q,d)p + g_{32}(c,w,p,q,d)q + g_{33}(c,w,p,q,d)d]$$

$$\text{for } \ 0 \le \rho \le 1, \tau > 0,$$

$$v(\rho,\tau) = u(\rho,\tau) - \rho u(1,\tau) \qquad (2.11)$$

for $0 \le \rho \le 1, \tau > 0,$

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2) = \eta^2(\tau)\mathrm{h}(c,w,p,q,d) \quad \text{for } 0 \le \rho \le 1, \tau > 0, \tag{2.12}$$

$$u(0,\tau) = 0 \text{ for } \tau > 0, \tag{2.13}$$

$$\frac{\partial\eta(\tau)}{\partial\tau} = \eta(\tau)u(1.\tau) \quad \text{for } \tau > 0 \tag{2.14}$$

$$p(\rho,0) = p_0(\rho), q(\rho,0) = q_0(\rho), d(\rho,0) = d_0(\rho), \text{ for } 0 \le \rho \le 1, \tag{2.15}$$

$$\eta(0) = \eta_0,$$

$$f(c,p,q) = F(c,p,q), \; g(w,p,q) = G(w,p,q),$$

$$\bar{c} = \bar{C}, \bar{w} = \bar{W}, c_0(\rho) = C(\rho R_0), c_0(\rho) = W_0(\rho R_0),$$

$$p_0(\rho) = P_0(\rho R_0), q_0(\rho) = Q_0(\rho R_0), d_0(\rho) = D_0(\rho R_0),$$

$$\eta_0 = R_0. \tag{2.16}$$

Conversely, if (η,c,w,p,q,d,u) is a solution of (2.2)-(2.16) such that η(τ)>0 for τ≥0, then by making the change of variables

$$r = \eta(\tau),\; t = \int_0^\tau \eta^2(s)ds, \mathrm{R}(t) = \eta(t),\; C(r,t) = c(\rho,\tau),$$

$$W(r,t) = w(\rho,\tau),$$

$$P(r,t) = p(\rho,\tau), Q(r,t) = q(\rho,\tau),$$

$$D(r,t) = d(\rho,\tau),\; v(r,t) = \frac{u(\rho,\tau)}{\eta(\tau)}$$

Lemma 2.1:
Under the change of variables (2.1) or its inverse (2.17), the free boundary problem (1.15) – (1.27) is equivalent to initial-boundary value problem (2.2) – (2.16).

Remark 2.2:
From (2.12),

$u(\rho,\tau) = \frac{\eta^2(\tau)}{\rho^2}\int_0^\rho h(c(s,\tau),w(s,\tau),p(s,\tau),q(s,\tau),d(s,\tau)s^2ds)$ is obtained.

Then using (2.14)-(2.18),

$$\frac{\partial\eta(\tau)}{\partial\tau} = \eta^3(\tau)\int_0^1 h(c(s,\tau),w(s,\tau),p(s,\tau),q(s,\tau),d(s,\tau)s^2ds).$$

We cannot expect the solution of (2.2) – (2.16) exists for all 0, but since we make the change of variables,

$t = \int_0^\tau \eta^2(s)ds$ and $\tau = \int_0^t \frac{ds}{R^2(s)}$, one we can prove the solution of(2.2)-(2.16) exists actually for all τ ≥ 0.

SECTION -3 EXISTENCE OF A LOCAL SOLUTION
From the assumptions (A1)-(A4) in sec.1 and transformation (2.1) in sce.2
we verify the following conditions hold:

(B1) f, g and h are C^1 – smooth functions;
(B2) *gij* (i,*j*=1,2,3) are C^1 – smooth functions;
(B3) p_0,q_0 and d_0 are C^1 – smooth functions;
(B4) $c_o(|x|)$, $w_0(|x|) \in Dp(B_1)$ for some *p*>5.

$\frac{\partial\sigma}{\partial n} = -V_n$ on $\partial\Omega(t), t > 0$ istence and uniqueness of solution[5] to fixed point theorem. Let,

$$M_0 = \|(p_0,q_0,d_0)\|_{L^\infty};$$

$$P(x,0) = P_0(x), Q(x,0) = Q_0(x), D(x,0) = D_0(x) \quad w \le \overline{w}, |p| \le 2M_0,$$

for $x \in \Omega(0)$
$|q| \le 2M_0, |d| \le 2M_0, i,j$

$$B_0 = max\{|h(c,w,p,q,d)|: 0 \le c \le \bar{c}, 0 \le w \le \overline{w},$$
$$|p| \le 2M_0, |q| \le 2M_0, |d| \le 2M_0\}.$$

Now given *T*>0, we introduce a metric space (*XT*,*d*) as

$$X_T = \{(\eta(\tau), c(\rho,\tau), w(\rho,\tau), p(\rho,\tau)q(\rho,\tau),$$
$$d(\rho,\tau))(0 \le \rho \le 1, 0 \le \tau \le T): (\eta,c,w,p,q,d)$$

satisfying the following conditions (C1)-(C4)

(C1) $\eta \in C[0,1]$, $\eta(0) = \eta_0$ and $1/2\,\eta_0 \le \eta(\tau) \le 2\eta_0 (0 \le \tau \le T)$;

(C2) $c \in C([0,1]\times[0,T])$, $c(\rho,0) = c_0(\rho), c(1,\tau) = \bar{c}$ and $0 \le c(\rho,\tau) \le \bar{c}$ for $0 \le \rho \le 1, 0 \le \tau \le T$;

(C3) $w \in C([0,1]\times[0,T])$, $w(\rho,0) = w_0(\rho), w(1,\tau) = \bar{w}$ and $0 \le w(\rho,\tau) \le \bar{w}$ for $0 \le \rho \le 1, 0 \le \tau \le T$;

(C4) $p(\rho,\tau), q(\rho,\tau), d(\rho,\tau) \in C([0,1]\times[0,T])$, $p(\rho,0) = p_0(\rho), q(\rho,0) = q_0(\rho)$,
$d(\rho,0) = d_0(\rho)$ and $|p(\rho,\tau)| \le 2M_0, |q(\rho,\tau)| \le 2M_0, |d(\rho,\tau)| \le 2M_0$
for $0 \le \rho \le 1, 0 \le \tau \le T$.

The metric d in *XT is defined by*

$$d((\eta_1,c_1,w_1,p_1,q_1,d_1),(\eta_2,c_2,w_2,p_2,q_2,d_2))$$
$$= \|\eta_1-\eta_2\|_{L\infty} + \|c_1-c_2\|_{L\infty} +$$
$$\|w_1-w_2\|_{L\infty} + \|p_1-p_2\|_{\Box\infty} +$$
$$\|q_1-q_2\|_{\Box\infty} + \|d_1-d_2\|_{\Box\infty}$$

It is easy to see (X_T,*d*) is a complete metric space.
Given any (η,c,w,p,q,d) ∈ X_T, set

$$u(\rho,\tau) = \frac{\eta^2(\tau)}{\rho}\int_0^\rho h(c(s,\tau),w(s,\tau),p(s,\tau),q(s,\tau),d(s,\tau))s^2\,ds$$

$$v(\rho,\tau) = u(\rho,\tau) - \rho u(1,\tau),$$

$$\phi(\rho,\tau) = \eta^2(\tau)f(c(s,\tau),p(s,\tau),q(s,\tau)),$$

$$\varphi(\rho,\tau) = \eta^2(\tau)g(w(s,\tau),p(s,\tau),q(s,\tau)).$$

Consider the following problem for $(\tilde{\eta},\tilde{c},\tilde{w},\tilde{p},\tilde{q},\tilde{d})$:

$$\frac{\partial\tilde{\eta}}{\partial\tau} = \tilde{\eta}(\tau)u(1,\tau) \text{ for } 0 < \tau \le T, \tag{3.1}$$

$$\tilde{\eta}(0) = \eta_0 \tag{3.2}$$

$$\frac{\partial\tilde{c}}{\partial\tau} = \frac{D_1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\tilde{c}}{\partial\rho}\right) + u(1,\tau)\rho\frac{\partial\tilde{c}}{\partial\rho} - \phi(\rho,\tau)\tilde{c} \quad \text{for } 0 < \rho < 1, 0 < \tau \le T,$$

$$\frac{\partial\tilde{c}}{\partial\tau}(0,\tau) = 0, \tilde{c}(1,\tau) = \tilde{c} \text{ for } 0 < \tau \le T, \tag{3.3}$$

$$\frac{\partial}{\partial t}(P+Q+D) + div(P+Q+D)\overline{v} = \frac{1}{N}(PK_B(C) - K_R(D)) \times$$
$$(N - (P+Q+D)) \quad \text{for } x \in \Omega(t), t>0.$$

(3.3)

$$\frac{\partial\tilde{w}}{\partial\tau} = \frac{D_2}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\tilde{w}}{\partial\rho}\right) + u(1,\tau)\rho\frac{\partial\tilde{w}}{\partial\rho} - \varphi(\rho,\tau)\tilde{w} \quad \text{for} 0 < \rho < 1, 0 < \tau \le T,$$

$$\frac{\partial\tilde{w}}{\partial\tau}(0,\tau) = 0, \tilde{w}(1,\tau) = \tilde{w} \text{ for } 0 < \tau \le T,$$

$\tilde{w}(\rho,0) = w_0(\rho)$ for $0 \le \rho \le 1$,

$$\frac{\partial \tilde{p}}{\partial \tau} + v\frac{\partial \tilde{p}}{\partial \rho} = \eta^2[g_{11}(\tilde{c},\tilde{w},\tilde{p},\tilde{q},\tilde{d})\tilde{p} + g_{12}(\tilde{c},\tilde{w},\tilde{p},\tilde{q},\tilde{d})\tilde{q} + g_{13}(\tilde{c},\tilde{w},\tilde{p},\tilde{q},\tilde{d})\tilde{d} +]$$

for $0 \le \rho \le 1, 0 < \tau \le T$,

$$\frac{\partial \tilde{q}}{\partial \tau} + v\frac{\partial \tilde{q}}{\partial \rho} = \eta^2[g_{21}(\tilde{c},\tilde{w},\tilde{p},\tilde{q},\tilde{d})\tilde{p} + g_{22}(\tilde{c},\tilde{w},\tilde{p},\tilde{q},\tilde{d})\tilde{q} + g_{23}(\tilde{c},\tilde{w},\tilde{p},\tilde{q},\tilde{d})\tilde{d} +]$$

for $0 \le \rho \le 1, 0 < \tau \le T$,

$$\frac{\partial \tilde{d}}{\partial \tau} + v\frac{\partial \tilde{d}}{\partial \rho} = \eta^2[g_{31}(\tilde{c},\tilde{w},\tilde{p},\tilde{q},\tilde{d})\tilde{p} + g_{32}(\tilde{c},\tilde{w},\tilde{p},\tilde{q},\tilde{d})\tilde{q} + g_{33}(\tilde{c},\tilde{w},\tilde{p},\tilde{q},\tilde{d})\tilde{d} +]$$

for $0 \le \rho \le 1, 0 < \tau \le T$

$\tilde{p}(\rho,0) = p_0(\rho)$, $\tilde{q}(\rho,0) = q_0(\rho)$, $\tilde{d}(\rho,0) = d_0(\rho)$, for $0 \le \rho \le 1$.

We define a mapping $F: (\eta, c, w, p, q, d) \to (\tilde{\eta}, \tilde{c}, \tilde{w}, \tilde{p}, \tilde{q}, \tilde{d})$.

Next to prove that F is a contraction mapping from X_T to X_T provided T is sufficiently small.

STEP 1:

F maps X_T into itself. It is obvious that (4.1)-(4.2) has a unique solution $\tilde{\eta} \in C^1[0,T]$ and

$$g_{11}(C,W,P,Q,D) = [K_B(C) - K_Q(C) - K_A(C) - t_1G_1(W)] - \frac{1}{N}[K_B(C)P - K_RD],$$

$$g_{12}(C,W,P,Q,D) = K_P(C),$$

(for $0 \le \rho \le 1, \tau > 0$, 3.13)

From the fact that $\|h(c(\rho,\tau), w(\rho,\tau), p(\rho,\tau), q(\rho,\tau), d(\rho,\tau))\|_{L^\infty} \le B_0$,

$\frac{1}{2}\eta_0 < \eta(\tau) \le 2\eta_0$, and $\|u(1,\tau)\|_{L^\infty} \le \frac{4}{3}B_0\eta_0^2$, then

$\eta_0 exp\{\frac{-4}{3}B_0\eta_0^2T\} \le \tilde{\eta}(\tau) \le \eta_0 exp\{\frac{4}{3}B_0\eta_0^2T\}$ or $0 \le \tau \le T$.

So if T is sufficiently small such that exp $\{\frac{4}{3}B_0\eta_0^2T\} \le 2$

$\frac{1}{2}\eta_0 \le \tilde{\eta} \le 2\eta_0$ that implies η satisfies the condition (C1). Next we consider (3.3)-(3.5) and (3.6)-(3.8).

Since $c_0(|x|), w_0(|x|) \in Dp(B_1)$ for some $p > 5$,3.3)-(3.5) and (3.6)-(3.8) has a $\tilde{c}(|x|,\tau) \in W_p^{2,1}(Q_T)$ unique solution and $\tilde{w}(|x|,\tau) \in W_p^{2,1}(Q_T)$ respectively. According to the embedding theorem $W_p^{2,1}(Q_T) \to C^{\lambda,\frac{\lambda}{2}}(\bar{Q}_T)$ where $\lambda = 2 - \frac{5}{p}$:hen $\tilde{c}(|x|,\tau), \tilde{w}(|x|,\tau) \in C([0,1]\times[0,1])$. By applying the maximum principle $0 \le \tilde{C} \le \bar{C}$ and $0 \le \tilde{W} \le \bar{W}$. Furthermore, by (3.4), the embedding, $W_p^{2,1}(Q_T) \to C^{1+\lambda,\frac{1+\lambda}{2}}(\bar{Q}_T)$. With $\lambda = 1 - \frac{5}{p}$ then $\left\|\frac{\partial \tilde{c}}{\partial \rho}\right\|_{L^\infty} \le A(T)$, $\left\|\frac{\partial \tilde{w}}{\partial \rho}\right\|_{L^\infty} \le A(T)$. From above results, we know $\tilde{c}$ satisfies the condition (C2) and $\tilde{w}$ satisfies the condition (C3). Finally we consider (4.9)-(4.12). Since $v(\rho,\tau)$, $\tilde{c}(\rho,\tau)$, $\tilde{w}(\rho,\tau)$ are continuously differentiable, then from Lemma 3.3 we obtain that if we take T small enough, (3.9)-(3.12) has a unique classical solution satisfying $(\tilde{p},\tilde{q},\tilde{d}) \in C^1([0,1]\times[0,1])$

$|\tilde{p}| \le 2M_0, |\tilde{q}| \le 2M_0, |\tilde{d}| \le 2M_0$, for $0 \le \rho \le 1, 0 \le \tau \le T$. 3.15)

Furthermore, if T is small enough, $\left\|\left(\frac{\partial \tilde{p}}{\partial \rho}, \frac{\partial \tilde{q}}{\partial \rho}, \frac{\partial \tilde{d}}{\partial \rho}\right)\right\|_{L^\infty} \le 2M_1$for $0 \le \rho \le 1, 0 \le \tau \le T$ where $M_1 = \|p_0', q_0', d_0'\|_{L^\infty}$ere implies p ,q , and d satisfy the condition (C4).

Now for a sufficiently small T, $F: \to X_T \to X_T$ is well defined.

To obtain the desired result we need to prove $F: \to X_T \to X_T$ is a contraction mapping if T is further small enough.

STEP 2:

Let $(\eta_i, c_i, w_i, p_i, q, d_i) \in X_T (i = 1,2)$ set

$$u_i(\rho,\tau) = \frac{\eta_i^2(\tau)}{\rho}\int_0^\rho h(c_i(s,\tau), w_i(s,\tau), p_i(s,\tau), q_i(s,\tau), d_i(s,\tau))s^2 ds,$$

$$v_i(\rho,\tau) = u_i(\rho,\tau) - \rho u_i(1,\tau),$$

$$(\tilde{\eta}_i, \tilde{c}_i, \tilde{w}_i, \tilde{p}_i, \tilde{q}_i, \tilde{d}_i) = F(\eta_i, c_i, w_i, p_i, q_i, d_i)$$

$$d = d((\eta_1, c_1, w_1, p_1, q_1, d_1), (\eta_2, c_2, w_2, p_2, q_2, d_2)).$$

From $\|h(c_i(\rho,\tau), w_i(\rho,\tau), p_i(\rho,\tau), q_i(\rho,\tau), d_i(\rho,\tau))\|_{L^\infty} \le B_0$ and $\frac{1}{2}\eta_0 < \eta_i(\tau) \le 2\eta_0$, easily calculate $|u_1(\rho,\tau) - u_2(\rho,\tau)| \le A(T)d$.

(3.17)

Then by (3.13) $\|\tilde{\eta}_1 - \tilde{\eta}_2\|_{L^\infty} \le \max_{0\le\tau\le T}|\tilde{\eta}_1(\tau) - \tilde{\eta}_2(\tau)| \le TA(T)d$. (3.18)

Next, let $\tilde{c}_* = \tilde{c}_1 - \tilde{c}_2$ and $w_* = \tilde{w}_1 - \tilde{w}_2$,we have

$$\frac{\partial \tilde{c}_*}{\partial \tau} = \frac{D_1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial \tilde{c}_*}{\partial \rho}\right) + u_1(\rho,\tau)\rho\frac{\partial \tilde{c}_*}{\partial \rho} - \phi(\rho,\tau)\tilde{c}_* + F(\rho,\tau)$$

for $0 < \rho < 1, 0 < \tau \le T$, (3.19)

$\frac{\partial \tilde{c}_*}{\partial \tau}(0,\tau) = 0, \tilde{c}_*(1,\tau) = 0$ for $0 \le \tau \le T$, (3.20)

$\tilde{c}_*(\rho,0) = 0$ for $0 \le \rho \le 1$,

$$\frac{\partial \tilde{w}_*}{\partial \tau} = \frac{D_2}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial \tilde{w}_*}{\partial \rho}\right) + u_1(\rho,\tau)\rho\frac{\partial \tilde{w}_*}{\partial \rho} - \phi(\rho,\tau)\tilde{w}_* + F(\rho,\tau)$$

for $0 < \rho < 1, 0 < \tau \le T$, (3.22)

$\frac{\partial \tilde{w}_*}{\partial \tau}(0,\tau) = 0, \tilde{w}_*(1,\tau) = 0$ for $0 \le \tau \le T$, (3.24)

where

$$\phi(\rho,\tau) = \eta_1^2(\tau)f(c_1(s,\tau), p_1(s,\tau), q_1(s,\tau)),$$

$$\varphi(\rho,\tau) = \eta_1^2(\tau)g(w_1(s,\tau), p_1(s,\tau), q_1(s,\tau)),$$

$$F(\rho,\tau) = [u_1(1,\tau) - u_2(1,\tau)]\rho\frac{\partial \tilde{c}_2}{\partial \rho} + [\eta_2^2(\tau)f(c_2, p_2, q_2) - \eta_1^2(\tau)f(c_1, p, q_1)]\tilde{c}_2,$$

$$G(\rho,\tau) = [u_1(1,\tau) - u_2(1,\tau)]\rho\frac{\partial \tilde{w}_2}{\partial \rho} + [\eta_2^2(\tau)g(w_2, p_2, q_2) - \eta_1^2(\tau)g(w_1, p, q_1)]\tilde{w}_2,$$

As for $\tilde{c}$, we know $\left\|\frac{\partial \tilde{c}_2}{\partial \rho}\right\|_{L^\infty} \le A(T)$ and $0 \le \tilde{c}_2(\rho,\tau) \le \bar{c}$, b maximum principle note that f is continuously differentiable and η_i, p_i, q_i are bounded ,so we can deduce that

$$\|F\|_{L^\infty} \le A(T)\|u_1 - u_2\|_{L^\infty} + \|\eta_2^2 f(c_2, p_2, q_2) - \eta_1^2 f(c_1, p, q_1)\|_{L^\infty} \le A(T)d.$$

$$\|\tilde{c}_1-\tilde{c}_2\|_{L^\infty}=\|\tilde{c}_*\|_{L^\infty}\le T\|F\|_{L^\infty}\le TA(T)d.$$

Similarly for $\tilde{w}$, we obtain

$$\|G\|_{L^\infty}\le A(T)\|u_1-u_2\|_{L^\infty}+A(T)d\le A(T)d.$$

Again we obtain

$$\|\tilde{w}_1-\tilde{w}_2\|_{L^\infty}=\|\tilde{w}_*\|_{L^\infty}\le T\|G\|_{L^\infty}\le TA(T)d.$$

Finally , letting $\tilde{p}^*=\tilde{p}_1-\tilde{p}_2$, $\tilde{q}^*=\tilde{q}_1-\tilde{q}_2$, $\tilde{d}^*=\tilde{d}_1-\tilde{d}_2$, then result is

$$\frac{\partial\tilde{q}_*}{\partial\tau}+v_1\frac{\partial\tilde{q}_*}{\partial\rho}=\lambda_{21}(\rho,\tau)\tilde{p}_*+\lambda_{22}(\rho,\tau)\tilde{q}_*+\lambda_{23}(\rho,\tau)\tilde{d}_*+F_2(\rho,\tau)$$

for $0\le\rho\le 1$,$0<\tau\le T$,

$$\frac{\partial\tilde{d}_*}{\partial\tau}+v_1\frac{\partial\tilde{d}_*}{\partial\rho}=\lambda_{31}(\rho,\tau)\tilde{p}_*+\lambda_{32}(\rho,\tau)\tilde{q}_*+\lambda_{33}(\rho,\tau)\tilde{d}_*+F_3(\rho,\tau)$$

for $0\le\rho\le 1$,$0<\tau\le T$,

$$\tilde{p}_*(\rho,0)=0,\ \tilde{q}_*(\rho,0)=0,\ \tilde{d}_*(\rho,0)=0,\ \text{for } 0\le\rho\le 1 \qquad (3.32)$$

where $\lambda_{ij}=\eta_1^2(\tau)g_{ij}(\tilde{c}_1,\tilde{w}_1,\tilde{p}_1,\tilde{q}_1,\tilde{d}_1)$ (i.j=1,2,3),

$$F_i(\rho,\tau)=(v_2-v_1)\frac{\partial\tilde{\xi}_i}{\partial\rho}+\sum_{j=1}^{3}\begin{pmatrix}\eta_1^2 g_{ij}(\tilde{c}_1,\tilde{w}_1,\tilde{p}_1,\tilde{q}_1,\tilde{d}_1)\\-\eta_2^2 g_{ij}(\tilde{c}_2,\tilde{w}_2,\tilde{p}_2,\tilde{q}_2,\tilde{d}_2)\end{pmatrix}\tilde{\xi}_j,$$

and $\tilde{\xi}_{1,}=\tilde{p}_2,\tilde{\xi}_{2,}=\tilde{q}_2,\tilde{\xi}_{3,}=\tilde{d}_2$. From (3.15)-(3.16) we know that

$$\|\tilde{p}_i\|_{L^\infty}\le 2M_0\ .\ \|\tilde{q}_i\|_{L^\infty}\le 2M_0\ ,\ \|\tilde{d}_i\|_{L^\infty}\le 2M_0\ ,\ (\text{i}=1,2),$$

$$\left\|\left(\frac{\partial\tilde{p}_i}{\partial\rho},\frac{\partial\tilde{q}_i}{\partial\rho},\frac{\partial\tilde{d}_i}{\partial\rho}\right)\right\|_{L^\infty}\le 2M_1\ ,\ (\text{i}=1,2),$$

and since *gij* (i,*j*=1,2,3) are continuously differentiable , we deduce that

$$\|F_i\|_{L^\infty}\le A(T)\|v_1-v_2\|_{L^\infty}+A(T)\sum_{j=1}^{3}\|\eta_1^2 g_{ij}(\tilde{c}_1,\tilde{w}_1,\tilde{p}_1,\tilde{q}_1,\tilde{d}_1)$$

$$-\eta_2^2 g_{ij}(\tilde{c}_2,\tilde{w}_2,\tilde{p}_2,\tilde{q}_2,\tilde{d}_2)\|_{L^\infty}$$

$\le A(T)\text{d}$, i=1,2,3 It is easy to see λ_ij (i,j=1,2,3) are bounded by a constant independent of the choice of (ηi,c_i,p_i,q_i,d_i) so from (3.33) we have

$$\|\tilde{p}_1-\tilde{p}_{2,}\tilde{q}_1-\tilde{q}_{2,}\tilde{d}_1-\tilde{d}_{2,}\|_{L^\infty}=\|\tilde{p}_*,\tilde{q}_*,\tilde{d}_*\|_{L^\infty}\le TA(T)\text{d}. \qquad (3.34)$$

By (3.16),(3.26).(3.28) and (3.34)

$$d\left(\eta_1,\tilde{c}_1,\tilde{w}_1,\tilde{p}_1,\tilde{q}_1,\tilde{d}_1\right),\left(\eta_2,\tilde{c}_2,\tilde{w}_2,\tilde{p}_2,\tilde{q}_2,\tilde{d}_2\right)\le TA(T)<1,$$

then F is a contraction mapping from X_T into X_T.

According to Banach fixed point theorem, if T is small enough then F has a unique fixed point (η, c, w, p, q, d) for $0\le\tau\le T$. By the definition of the mapping F (, c, w, p, q, d) is the unique solution of the problem (2.2) – (2.16) for for $0\le\tau\le T$

THEOREM 4.1:

Under the assumptions (A1) – (A4) and initial condition (1.30), the free boundary problem (1.15)-(1.27) has a unique solution (*R,C,W,P,Q,D*) for all

In addition, for any *T>0, R(t)*∈C^1 *[0,T],C,W*∈$W_p^{2,1}$ *(Q_T^R)* and P,Q,D∈C^1 (Q_T^R)

Furthermore, the following estimates hold:

$R(t)>0$ for >0 ,

$0<C(r,t)\le\bar{C}$, $0<W(r,t)\le\bar{W}$ for $0\le r\le R(t)$, t≥ 0,

$P(r,t)\ge 0$, $Q(r,t)\ge 0$, $D(r,t)\ge 0$ for $0\le r\le R(t)$, t≥ 0,

$P(r,t)+Q(r,t)+D(r,t)=N$ for $0\le r\le R(t)$, t≥ 0

there exists T > 0 depending only on

$$\|c_0(|x|)\|_{Dp(B_{R_0})},\|w_0(|x|)\|_{Dp(B_{R_0})},\|p_{0,}q_{0,}d_{0,}\|_{L^\infty,}\|(p_{0,}'q_{0,}'d_0')\|_{L^\infty},$$

such that the problem (2.2)-(2.16) has a unique solution for 0 ≤ τ ≤T.

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